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THE MOTION OF A POINT MASS ALONG A VIBRATING STRING†

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The Cauchy problem for a system of equations which described the undetached motion of a point mass along a string is considered. The problem can be reduced to an initial-value problem for two integro-differential equations. If the initial velocities of the points of the string are zero, the problem reduces to the Cauchy problem for a system of three first-order ordinary differential equations. The problem with zero initial displacements is much simpler. The case of an impact along the string is considered in detail.

The essence of the concepts of wave pressure, wave momentum and so on becomes very clear from an analysis of the consistent motion of objects along elastic strings. While the problem of the motion of a point mass along an infinite vibrating string is easy to formulate, it involves the main difficulties of problems of this kind: variable boundaries and non-linearity.

1. A point mass is a mathematical idealization of a sphere of small radius with a narrow channel (a "bead") through which the string passes without friction. The equations which describe this dynamical system have the form [1–5]

$$V_{tt} = c^2 V_{xx} = 0, \quad c = \sqrt{N/\rho} \tag{1.1}$$

$$m\ddot{V}(t,l(t)) = \rho(c^2 - \dot{l}^2)[V_x], \quad m\ddot{l} = -\frac{\rho}{2}(c^2 - \dot{l}^2)[V_x^2]$$

The initial conditions are assumed to be given on the entire real axis

$$V(0,x) = \varphi(x), \quad V_t(0,x) = \psi(0,x) \tag{1.2}$$

where N is the tension in the string, ρ is the linear density, square brackets are used to denote the difference between the limiting values on the right and left of the boundary $x = l(t)$ of the expression that they contain, $V(t, l(t))$ is the transverse deviation of the bead, $l(t)$ is its longitudinal displacement along the string, a dot above a symbol denotes an ordinary derivative with respect to time, and the subscripts t and x denote the partial derivatives with respect to time and the space variable respectively. To (1.1) and (1.2) we add the following initial conditions for a point mass

$$l(0) = l_0, \quad \dot{l}(0) = w, \quad V(0,l_0) = \varphi(l_0), \quad V_t(0,l_0) = \psi(l_0)$$

2. The computations can be simplified by changing to the characteristic variables $\xi = x - ct$, $\eta = x + ct$. Suppose that the longitudinal displacement of the bead is mapped onto the plane of the characteristic variables and that the coordinates of the bead in that plane at time t are ξ, η (the point $P(\xi, \eta)$ in Fig. 1). From $P(\xi, \eta)$ we draw the characteristics of the wave equation as far as the intersection with the straight line $\eta = \xi$ on which the initial conditions are given. Using Gauss' formula

$$\iint_D V_{\xi\eta} d\xi d\eta = \frac{1}{2}\oint_{\partial D} (V_\eta d\eta - V_\xi d\xi)$$

after integrating along the contours of figures APC and CPB (Fig. 1), returning to the original variables x, t and c and taking the initial conditions (1.2) into account we obtain

$$c\int_0^t V_x^- dt + \frac{1}{c}\int_{l_0}^{l(t)} V_t^- dx = V(t,l(t)) - \varphi(l - ct) - \frac{1}{c}\int_{l-ct}^{l_0} \psi(x)dx$$

$$c\int_0^t V_x^+ dt + \frac{1}{c}\int_{l_0}^{l(t)} V_t^+ dx = -V(t,l(t)) + \varphi(l + ct) + \frac{1}{c}\int_{l_0}^{l+ct} \psi(x)dx \tag{2.1}$$

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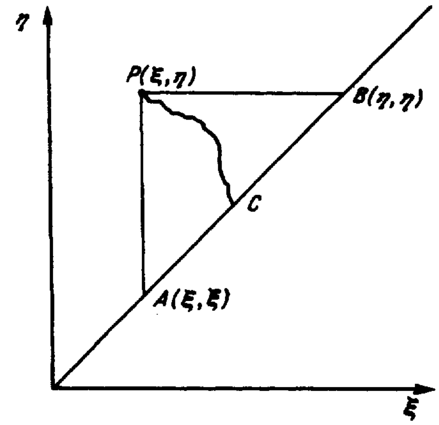

Fig. 1.

where a minus superscript denotes that the limiting value is taken on the left, while a plus superscript denotes that it is taken on the right of the curve CP.

The fact that the string does not break at the bead can be expressed by the formula

$$V_t^+ + V_x^+ \dot{l} = V_t^- + V_x^- \dot{l} \tag{2.2}$$

Using expression (2.2) and subtracting the first relation of (2.1) from the second, we obtain

$$\int_0^t \left(c - \frac{\dot{l}^2}{c}\right)[V_x]dt = -2V(l,t) + \varphi(l+ct) + \varphi(l-ct) + \frac{1}{c}\int_{l-ct}^{l+ct} \psi(x)dx \tag{2.3}$$

We now integrate the second equation of (1.1) once, and replace the integral on its right-hand side by the right-hand side of (2.3). As a result, we can express this integral in terms of functions of the initial condition and $V(l, t)$. We obtain

$$\dot{V} + \frac{2\rho c}{m} V = \frac{\rho c}{m}\left\{\varphi(l+ct) + \varphi(l-ct) + \frac{1}{c}\int_{l-ct}^{l+ct} \psi(x)dx\right\} \tag{2.4}$$

The transverse deviation of the point mass is easily found from (2.4) when the longitudinal displacement is known.

We now derive an equation for the longitudinal displacement. Differentiating Eqs (2.1) with respect to time and then adding them using formula (2.2), we obtain

$$\left(c - \frac{\dot{l}^2}{c}\right)(V_x^+ + V_x^-) = -\frac{2\dot{l}}{c}\dot{V}(l,t) + \Phi^+(\dot{l},l,t) + \Phi^-(\dot{l},l,t) \tag{2.5}$$

$$\Phi^{\pm}(\dot{l},l,t) = \pm\varphi'(l \pm ct)(\dot{l} \pm c) + \frac{1}{c}\psi(l \pm ct)(\dot{l} \pm c)$$

where the prime denotes the ordinary derivative.

We now differentiate Eq. (2.3) with respect to time, multiply its left- and right-hand sides by the respective parts of Eq. (2.5), and then multiply by $\rho/[2(c^2 - \dot{l}^2)]$. The left-hand side of the resulting equation is the same as the right-hand side of the last equation of system (1.1), which describes the longitudinal displacement of the bead. Hence, for the longitudinal acceleration of the point mass we have the expression

$$\ddot{l} = -\frac{\rho c^2}{2m(c^2 - \dot{l}^2)}\{-2\dot{V}(l,t) + \Phi^+(\dot{l},l,t) - \Phi^-(\dot{l},l,t)\}\left\{-\frac{2\dot{l}}{c}\dot{V}(l,t) + \Phi^+(\dot{l},l,t) + \Phi^-(\dot{l},l,t)\right\} \tag{2.6}$$

Thus we have a system of two related equations (2.4) and (2.6) from which to find the longitudinal and transverse displacements of the bead.

3. We note that if the string is initially released without communicating velocities to its points, a system of two ordinary differential equations is obtained. If the dimensionless number $\rho c T_0/m$ is small (T_0 is the characteristic

time), the acceleration of the point mass is described approximately by the expression

$$\ddot{l} \approx -\frac{\rho}{2m}\{c^2\varphi'^2(\chi)-c^2\varphi'^2(\zeta)+\psi^2(\chi)-\psi^2(\zeta)+2c\varphi'(\chi)\psi(\chi)+2c\varphi'(\zeta)\psi(\zeta)\},$$

$$\chi=l+ct,\quad \zeta=l-ct$$

which, for zero initial velocities of points of the string in particular, takes the simple form

$$\ddot{l} \approx -\frac{\rho c^2}{2m}\{\varphi'^2(\chi)-\varphi'^2(\zeta)\} \tag{3.1}$$

Some difficulty arises in using (3.1) to find displacements of the bead for small t, owing to the fact that its right-hand side contains an implicit function of l.

4. The system of equations (2.4) and (2.6) becomes much simpler in the rather interesting case of a hammer blow along the segment $[x_1, x_2]$ (to fix our ideas, we will assume that this segment lies to the left of the bead). In this case the initial conditions take the form

$$V(0,x)=0,\quad V_t(0,x)=\psi_0 H(x-x_1)H(x_2-x)$$

where $H(x)$ is the Heaviside unit step function. The system takes the form

$$\dot{V}+\frac{2\rho c}{m}V=\frac{\rho}{m}\psi_0 H(x_2-\zeta)\{x_2-\zeta H(\zeta-x_1)-x_1H(x_1-\zeta)\},\quad \zeta=l-ct$$

$$\ddot{l}=-\frac{\rho c^2}{2m(c^2-\dot{l}^2)}\left\{\frac{4\dot{l}}{c}\dot{V}^2-\frac{\psi_0^2}{c^2}H(\zeta-x_1)H(x_2-\zeta)(\dot{l}-c)^2\right\}$$

Figure 2 shows the relation between the velocity of the bead $\dot{l}/c$ and dimensionless time $\tau = tc/(x_2-x_1)$ for different values of $D = \rho(x_2-x_1)/m$, when the other dimensionless term $E = \psi/c$ is constant, equal to 0.5. If $D < 1$ the velocity of the bead gradually increases, approaching a certain constant value. The reason for this is that once the wave has caught up with the point mass, it ceases to act on it. The bead velocity becomes constant at a value of D of unity or more, but non-monotonely: there is a small segment of drag at the trailing edge of the wave. This is rather interesting, if the dynamic interaction of the bead and string is explained in terms of the popular concept of wave pressure. Figure 3 possibly makes this even clearer. The four curves show how the bead gathers speed when the parameter $D = 1$ is fixed and E takes different values. The drag segment is very evident on the curve for $E = 2$. It can be attributed to a kink in the derivative of V with respect to x on the trailing edge of the wave, where for a short time the difference of the squares of the derivatives to the right and left becomes positive.

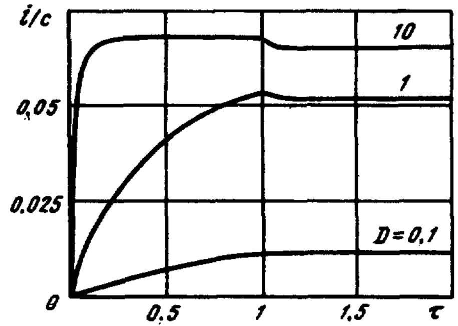

Fig. 2.

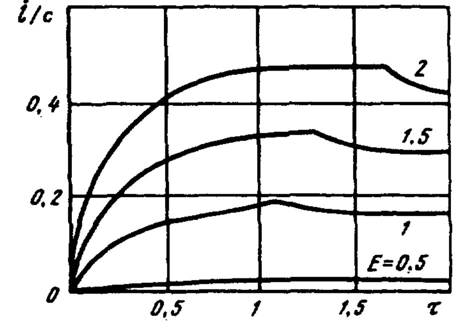

Fig. 3.

REFERENCES

1. KAZHAYEV, V. V. and UTKIN, G. A., The motion of a mass along a string under the effect of wave pressure. In *Differential and Integral Equations*. Gor'k. Gos. Univ., Gorky, 1989, 112–117.
2. VESNITSKII, A. I. and UTKIN, G. A., The motion of a body along a string under the effect of wave pressure. *Dokl. Akad. Nauk SSSR*, 1988, **302**, 278–280.
3. VESNITSKII, A. I., KAPLAN, L. Ye. and UTKIN, G. A., Laws and changes of energy and momentum for one-dimensional systems with moving clampings and loads. *Prikl. Mat. Mekh.*, 1983, **7**, 863–866.
4. KOSHELEV, S. V. and UTKIN, G. A., The existence, uniqueness and some properties of the solution of the problem of a bead on a string. In *Wave Problems of Mechanics*, Izd. Nizhegorod. Univ., Nizhnii Novgorod, 1992, 36–48.
5. UTKIN, G. A., Formulation of problems of the dynamics of elastic systems and objects that move along them. In *Wave Dynamics of Machines*. Nauka, Moscow, 1991, 4–14.

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